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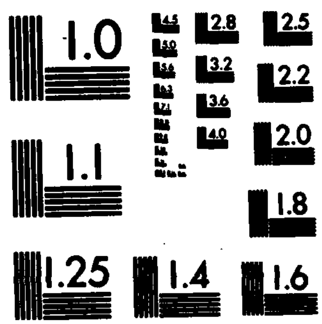

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ON A GENERAL APPROACH TO BIBD

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November 1982

Technical Report No. 82-34

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1. INTRODUCTION

At first we introduce some notations.

If S is a finite set with cardinal $|S| = \alpha$, we will use $\sum_\beta(S)$ or $\sum_\beta(\alpha)$ to denote the set of all β-subsets of S, here $1 \leq \beta \leq \alpha$, and β-subset means subset with cardinal β.

Let A_β (or $A_\beta(S)$, or $A_\beta(\alpha)$) be a function from $S \times \sum_\beta$ into the 2-set $\{0,1\}$, defined by

$$A_\beta(\xi,\eta) = \begin{cases} 1, & \text{if } \xi \in \eta, \\ 0, & \text{otherwise,} \end{cases}$$

here $\xi \in S$, $\eta \in \sum_\beta$.

In the same way, we define a function B_β (or $B_\beta(S)$, or $B_\beta(\alpha)$) from $\sum_2 \times \sum_\beta$ into $\{0,1\}$ by

$$B_\beta(\xi,\eta) = \begin{cases} 1, & \text{if } \xi \subset \eta \\ 0, & \text{otherwise,} \end{cases}$$

here $\xi \in \sum_2$ and $\eta \in \sum_\beta$.

The function $B_\beta(\alpha)$ is important in the theory of BIBD (about the definition of BIBD see M. Hall [1]). A BIBD with parameters v,b,r,k,λ can be represented as a function x defined on $\sum_k(v)$, and for $\eta \in \sum_k(v)$, $x(\eta)$ = the number of times N occurs in this design. It is easy to see that a nonnegative integral valued function x on $\sum_k(v)$ represents a BIBD (v,b,r,k,λ) if and only if

$$(1) \quad \sum_{\eta \in \Sigma_k} B_k(\xi,\eta)x(\eta) = \lambda, \qquad \xi \in \textstyle\sum_2,$$

and $bk = rv$, $\lambda(v-1) = r(k-1)$.

Hedayat and Li in [2] introduced a notion called trade. An integral-valued function x defined on Σ_k is called a trade if

$$(2) \quad \sum_{\eta\epsilon\Sigma_k} B_k(\xi,\eta)x(\eta) = 0, \qquad \xi\epsilon\Sigma_2.$$

They indicated that to construct trades is important and difficult.

In this article, we find a nonsingular matrix P such that PB_k is of triangular form. The existence of P is well-known, we get a concrete P. Thus the work to solve equations (1) and (2) might be made easier.

Acknowledgement. The author is grateful to Professor O. Kempthorne for his comments.

2. BLOCK DECOMPOSISION OF $B_k(v)$.

The key to solve the reduction problem is to decompose the matrix $B_k(v)$ into appropriate blocks. In this section we will describe the block decomposition we use.

At first we introduce some notations.

Let $k_1 = k-2$, $k_2 = v-k_1 = v-k+2$. We assume $k<v-1$, then $k_2 \geq 4$. Let

$X = \{1,2,\ldots,k_1\}$,

$Y = \{k_1+1,\ldots,k_1+k_2\}$,

$X_i = X \setminus \{i\}$, $\quad i=1,\ldots,k_1$,

$X_{ij} = X \setminus \{i,j\}$, $\quad i,j$ are different and in X,

...

If $\underline{F}=\{F\}$, $\underline{G}=\{G\}$ are two families of sets of integers, $\underline{F} \cup \underline{G}$ will denote the family of all sets of the form $F \cup G$, with $F \in \underline{F}$ $G \in \underline{G}$. If $\underline{F}$ contains only one set F, $F \cup \underline{G}$ will denote $\underline{F} \cup \underline{G}$.

At first we describe how to arrange the rows and how to decompose the rows into groups.

The rows corresponding to the sets of $\sum_2(v)$. We give $\sum_2(v)$ an order, that is the lexicographical order, and then decompose $\sum_2(v)$ into the following disjoint sets:

$$\textstyle\sum_2(X),\ \{1\} \cup \sum_1(Y), \ldots, \{k_1\} \cup \sum_1(Y),\ \sum_2(Y).$$

Now we describe the arrangement and the decomposition of the columns of $B_k(v)$.

The columns of the matrix $B_k(v)$ correspond to the sets in $\Sigma_k(v)$. The order of $\Sigma_k(v)$ is also the lexicographical one, and the decomposition of columns is according to the following decomposition of the set $\Sigma_k(v)$:

$$\Sigma_1(X)\,\dot\cup\,\Sigma_2(Y);\; X_{k_1}\,\dot\cup\,\Sigma_3(Y),\ldots,X_1\,\dot\cup\,\Sigma_3(Y);\; X_{k_1-1,k_1}\,\dot\cup\,\Sigma_4(Y),\ldots,X_{1,2}\,\dot\cup\,\Sigma_4(Y);$$

$$X_{k_1-2,k_1-1,k_1}\,\dot\cup\,\Sigma_5(Y),\ldots;\ldots$$

Thus, the matrix $B_k(v)$ is decomposed into the following block matrix.

$\Sigma_2(X)$	J	K_{J_1}	K_{k_1-1}	$\cdots$	K_2	K_1	K_{k_I1,k_1}	K_{k_I2,k_1}	$\cdots$	$K_{1,3}$	K_{12}	K_{k_I2,k_I1,k_1}	$\cdots$
$\{1\}\dot\cup\Sigma_1(Y)$	A_2	A_3	A_3	$\cdots$	A_3	0	A_4	A_4	$\cdots$	0	0	A_5	$\cdots$
$\{2\}\dot\cup\Sigma_1(Y)$	A_2	A_3	A_3	$\cdots$	0	A_3	A_4	A_4	$\cdots$	A_4	0	A_5	$\cdots$
$\{3\}\dot\cup\Sigma_1(Y)$	A_2	A_3	A_3	$\cdots$	A_3	A_3	A_4	A_4	$\cdots$	0	A_4	A_5	$\cdots$
$\cdots$	$\cdots$												
$\{k_I2\}\dot\cup\Sigma_1(Y)$	A_2	A_3	A_3	$\cdots$	A_3	A_3	A_4	0	$\cdots$	A_4	A_4	0	$\cdots$
$\{k_I1\}\dot\cup\Sigma_1(Y)$	A_2	A_3	0	$\cdots$	A_3	A_3	0	A_4	$\cdots$	A_4	A_4	0	$\cdots$
$\{k_1\}\dot\cup\Sigma_1(Y)$	A_2	0	A_3	$\cdots$	A_3	A_3	0	0	$\cdots$	A_4	A_4	0	$\cdots$
$\Sigma_2(Y)$	I	B_3	B_3	$\cdots$	B_3	B_3	B_4	B_4	$\cdots$	B_4	B_4	B_5	$\cdots$
	$X\dot\cup\Sigma_2(Y)$	$X_{k_1}\dot\cup\Sigma_3(Y)$	$X_{k_I1}\dot\cup\Sigma_3(Y)$	$\cdots$	$X_2\dot\cup\Sigma_3(Y)$	$X_1\dot\cup\Sigma_3(Y)$	$X_{k_I1,k_1}\dot\cup\Sigma_4(Y)$	$X_{k_I2,k_1}\dot\cup\Sigma_4(Y)$	$\cdots$	$X_{1,3}\dot\cup\Sigma_4(Y)$	$X_{1,2}\dot\cup\Sigma_4(Y)$	$X_{k_I2,k_I1,k_1}\dot\cup\Sigma_5(Y)$	$\cdots$

Here, in this matrix of blocks, $A_2=A_2(k_2)$, $A_3=A_3(k_2)$,...., $B_3=B_3(k_2)$, $B_4=B_4(k_2)$,...; J is a matrix with all its entries equal to 1; matrices $K_1,\ldots,K_{k_1},K_{1,2},K_{1,3},\ldots$have constant rows.

3. FIRST REDUCTION

We will apply elementary operations on the block matrix of 2.

At first we prove

Lemma 1. $A_2 B_\ell = (\ell-1)A_\ell$.

Proof. For two sets E and F we define

$$1_F(E) = \begin{cases} 1, & \text{if } E \subset F, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the entry of $A_2 B_\ell$ at i-th row and F-th column ($i\varepsilon\{1,2,\ldots,k_2\}$, $F\varepsilon\sum_\ell(k_2)$) would be

$$\sum_{E\varepsilon\Sigma_2(k_2)} 1_E(i)1_F(E).$$

A summand in this sum is not 0 iff

$i\varepsilon E\subset F$.

If i does not belong to F, this relation cannot hold for any $E\varepsilon\sum_2(k_2)$, and the sum is 0. But if i belongs to F, $F\setminus\{i\}$ contains $\ell-1$ numbers, each of which and i constitute a set $E\varepsilon\sum_2(k_2)$ with the property that $i\varepsilon E\subset F$. Thus the sum is $\ell-1$. Summing up, we have

$$\sum_{E\varepsilon\Sigma_2(k_2)} 1_E(i)1_F(E) = (\ell-1)1_F(i),$$

i.e. $A_2 B_\ell = (\ell-1)A_\ell$.

Lemma 2. $JB_\ell = \binom{\ell}{2}J$, here J is the matrix with all entries 1.

Pf: Take any $F \in \sum_\ell(k_2)$, the F-th component of the row vector $[1,\ldots,1]\ B_\ell$ is

$$\sum_{E \in \Sigma_2(k_2)} 1_F(E),$$

it is the number of 2-subsets of E, i.e. $\binom{\ell}{2}$.

<u>Reduction</u> 1: From each k_1 intermediate rows of blocks, subetract A_2 times the last row.

From the first row of blocks, subetract J times the last row.

By Lemma 1 and 2, we would get a matrix of the form.

$$
\begin{bmatrix}
0 & C_{k_1} & C_{k_1-1} & & C_2 & C_1 & C_{k_1-1,k_1} & C_{k_1-2,k_1} & & C_{1,3} & C_{1,2} & C_{k_1-2,k_1-1,k_1} & \\
0 & -A_3 & -A_3 & \vdots & -A_3 & -2A_3 & -2A_4 & -2A_4 & \vdots & -3A_3 & -3A_4 & -3A_5 & \vdots \\
0 & -A_3 & -A_3 & & -2A_3 & -A_3 & -2A_4 & -2A_4 & & -2A_4 & -3A_4 & -3A_5 & \\
0 & -A_3 & -A_3 & & -A_3 & -A_3 & -2A_4 & -2A_4 & & -3A_4 & -2A_4 & -3A_5 & \\
 & \cdots & & & & & \cdots & & & & \cdots & & \\
0 & -A_3 & -A_3 & & -A_3 & -A_3 & -2A_4 & -3A_4 & & -2A_4 & -2A_4 & -4A_5 & \\
0 & -A_3 & -2A_3 & \vdots & -A_3 & -A_3 & -3A_4 & -2A_4 & \vdots & -2A_4 & -2A_4 & -4A_5 & \vdots \\
0 & -2A_3 & -A_3 & & -A_3 & -A_3 & -3A_4 & -3A_4 & & -2A_4 & -2A_4 & -4A_5 & \\
I & B_3 & B_3 & & B_3 & B_3 & B_4 & B_4 & & B_4 & B_4 & B_5 &
\end{bmatrix}
$$

Here, by Lemma 2,

$$
\begin{aligned}
C_i &= K_i - JB_3 &&= K_i - \binom{3}{2}J &&= K_i - 3J, \\
C_{ij} &= K_{ij} - JB_4 &&= K_{ij} - \binom{4}{2}J &&= K_{ij} - 6J, \\
C_{ij\ell} &= K_{ij\ell} - JB_5 &&= K_{ij\ell} - \binom{5}{2}J &&= K_{ij\ell} - 10J,
\end{aligned}
$$

are matrices with constant rows.

<u>Reduction 2°</u>. Take the sume of the k_1 intermediate rows of blocks to be the new second row of blocks, and divide this row by (k_1+1), because

$$r(k_1-r) + (r+1)r = r(k_1+1)$$

the new second row becomes

$$[0,\ A_3,\ A_3,\ \ldots,\ A_3,\ A_3,\ 2A_4,\ 2A_4,\ \ldots,\ 2A_4,\ 2A_4,\ 3A_5,\ \ldots\ 3A_5,\ \ldots].$$

Add this row to the other k_1-1 intermediate rows, and add these rows (3^{rd}, ..., $k_1{}^{th}$) to the second row, and then multiply some rows by (-1), we get

$$
\begin{bmatrix}
0 & C_{k_1} & C_{k_1-1} & & C_2 & C_1 & C_{k_1-1,k_1} & C_{k_1-2,k_1} & & C_{1,3} & C_{1,2} & C_{k_1-2,\,k_1-1,\,k_1} & \\
0 & 0 & 0 & & 0 & A_3 & 0 & 0 & & A_4 & A_4 & 0 & \\
0 & 0 & 0 & \vdots & A_3 & 0 & 0 & 0 & \vdots & 0 & A_4 & 0 & \vdots \\
0 & 0 & 0 & & 0 & 0 & 0 & 0 & & A_4 & 0 & 0 & \\
0 & 0 & 0 & & 0 & 0 & 0 & A_4 & & 0 & 0 & A_5 & \\
0 & 0 & A_3 & & 0 & 0 & A_4 & 0 & & 0 & 0 & A_5 & \\
0 & A_3 & 0 & \vdots & 0 & 0 & A_4 & A_4 & \vdots & 0 & 0 & A_5 & \vdots \\
I & B_3 & B_3 & & B_3 & B_3 & B_4 & B_4 & & B_4 & B_4 & B_5 &
\end{bmatrix}
$$

__Lemma 3__. $C_iA_\ell = \ell C_i$

Proof. Because C_i is a matrix with constant rows, and the sum of entries on each column of A_ℓ is ℓ, thus $C_iA_\ell = \ell C_i$.

__Reduction 3°__. From the first row of blocks substract 1/3 time the sum of the k_1 intermediate rows. We get by Lemma 3,

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & D_{k_I1,k_1} & D_{k_I2,k_1} & D_{13} & D_{12} & D_{k_I2,k_I1,k_1} \\
0 & 0 & 0 & 0 & A_3 & 0 & 0 & A_4 & A_4 & 0 \\
0 & 0 & 0 & A_3 & 0 & 0 & 0 & 0 & A_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 \\
 & \cdots & & & & & \cdots & & \cdots & \\
0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & A_5 \\
0 & 0 & A_3 & 0 & 0 & A_4 & 0 & 0 & 0 & A_5 \\
0 & A_3 & 0 & 0 & 0 & A_4 & A_4 & 0 & 0 & A_5 \\
I & B_3 & B_3 & B_3 & B_3 & B_4 & B_4 & B_4 & B_4 & B_5
\end{bmatrix}$$

Here

$$D_{ij} = C_{ij} - \frac{1}{3}C_iA_4 - \frac{1}{3}C_jA_4 = C_{ij} - \frac{4}{3}(C_i+C_j)$$

$$D_{ij\ell} = C_{ij\ell} - \frac{1}{3}C_iA_5 - \frac{1}{3}C_jA_5 - \frac{1}{3}C_\ell A_5$$

$$= C_{ij\ell} - \frac{5}{3}(C_i+C_j+C_\ell),$$

$\cdots$

4. FURTHER REDUCTION

Now we consider the problem: How to reduce the matrix

$[D_{k_I 1,k_1}\ \ D_{k_I 2,k_1}\ \ \cdots\ \ D_{13}\ D_{12}]$.

From 3, we know that

$$\begin{aligned} D_{ij} &= C_{ij} - \tfrac{4}{3}(C_i + C_j) \\ &= K_{ij} - \tfrac{4}{3}K_i - \tfrac{4}{3}K_j + 2J. \end{aligned}$$

For a 2-set $\{p,q\} \varepsilon \sum_2(K_1)$, the row of D_{ij} corresponding to this set has entries

$$D_{pq,ij} = \begin{cases} 1 - \frac{4}{3} - \frac{4}{3} + 2 = \frac{1}{3}, & \text{if } \{p,q\} \cap \{i,j\} = \emptyset, \\ 0 - \frac{4}{3} + 2 = \frac{2}{3}, & \text{if } |\{p,q\} \cap \{i,j\}| = 1, \\ 0 + 0 + 0 + 2 = 2, & \text{if } \{p,q\} = \{i,j\}. \end{cases}$$

Lemma 4. If $H_n = (h_{pq,ij})$, $G_n = (g_{pq,\ ij})$ $1 \le p \le q \le n$, $1 \le i \le j \le n$, are two matrices, defined by

$$h_{pq,ij} = \begin{cases} 6, & \text{if } p = i,\ q = j; \\ 2, & \text{if } |\{p,q\} \cap \{i,j\}| = 1; \\ 1, & \text{if } \{p,q\} \cap \{i,j\} = \emptyset, \end{cases}$$

$$g_{pq,ij} = \begin{cases} 1, & \text{if } p = i,\ q = j; \\ -\frac{1}{n-p+1}, & \text{if } p < i \text{ and } q = i \text{ or } j; \\ 0, & \text{otherwise}. \end{cases}$$

then

$$G_nH_n = \begin{bmatrix} (a_1-b_1)I_{n-1}+b_1J_{n-1} & 0 & 0 & \cdots & 0 & 0 \\ & (a_2-b_2)I_{n-2}+b_2J_{n-2} & 0 & \cdots & 0 & 0 \\ & & (a_3-b_3)I_{n-3}+b_3J_{n-3} & \cdots & 0 & 0 \\ & & & & \vdots & \vdots \\ & & & & (a_{n-2}-b_{n-2})I_{n-2}+b_{n-2}J_{n-2} & 0 \\ & & & & & a_{n-1} \end{bmatrix}$$

here

$$a_i = b-2\,\frac{n-i-1}{n-i+1}\,, \qquad i=1,\ \ldots,\ n-1,$$

$$b_i = 2-\frac{n-i}{n-i+1}\,, \qquad i=1,\ \ldots,\ n-2.$$

Proof. Let $L = (L_{pq,ij}) = G_nH_n$. Then

$$L_{pq,ij} = \sum_{\mu<\nu} g_{pq,\mu\nu}h_{\mu\nu,ij} = h_{pq,ij} + \sum_{\nu>q} g_{pq,q\nu}h_{q\nu,ij} + \sum_{p<\mu<q} g_{pq,\mu q}h_{\mu q,ij}.$$

Case 1. $L_{pq,pq} = 6+(-\frac{1}{n-p+1})((n-q)2+(q-p-1)2)$

$$= 6-2\,\frac{n-p-1}{n-p+1}\,.$$

Case 2. $p=i$, $q<j$.

$$L_{pq,pj} = h_{pq,pj} + \sum_{\nu>q} g_{pq,q\nu}h_{q\nu,pj} + \sum_{p<\mu<q} g_{pq,\mu q}h_{\mu q,pj}$$

$$= 2-\frac{1}{n-p+1}(n-1-1+2+q-p-1) = 2-\frac{n-p}{n-p+1}\,.$$

Case 3. $p=i$, $q>j$. Similar to Case 2,

$$L_{pq,pj} = 2-\frac{n-p}{n-p+1}.$$

Case 4. $p<i$, $q=i$

$$L_{pq,qj} = h_{pq,qj} + \sum_{\nu>q} g_{pq,q\nu} h_{q\nu,qj} + \sum_{p<\mu<q} g_{pq,\mu q} h_{\mu q,qj}$$

$$= 2+(-\frac{1}{n-p+1})((n-q-1)2+6+2(q-p-1)) = 0.$$

Case 5. $p<i$, $q=j$. Similar to Case 4.

Case 6. $p<i$, $q\neq i$, $q\neq j$.

$$L_{pq,ij} = 1 - \frac{1}{n-p+1}(n-p-1+2) = 0.$$

Thus, the remaining problems are how to reduce $aI+bJ$ and A_3. But these problems can be solved with the help of the following lemmas.

Lemma 5. If $a\neq b$, then

$$[(a+(n-1)b)I-bJ][(a-b)I+bJ] = (a-b)(a+(n-1)b)I.$$

Proof. Direct verification.

Lemma 6. $A_\gamma(\alpha)$ can have the following form

$$A_\gamma(\alpha) = \begin{bmatrix} J_{(\gamma-1)\times(\alpha-\gamma+1)} & J_{(\gamma-1)\times(\gamma-1)}-I_{(\gamma-1)} & E \\ I_{\alpha-\gamma+1} & D & F \end{bmatrix}.$$

If

$$P = \begin{bmatrix} -\gamma I_{\gamma-1}+J_{(\gamma-1)\times(\gamma-1)} & J_{(\gamma-1)\times(\alpha-\gamma+1)} \\ 0 & I_{\alpha-\gamma+1} \end{bmatrix}$$

then

$$PA_\gamma(\alpha) = \begin{bmatrix} 0 & \gamma I_{\gamma-1} & E_1 \\ I_{\alpha-\gamma+1} & D & F \end{bmatrix}, \quad E_1 = -\gamma E+J_{(\gamma-1)\times(\gamma-1)}E+J_{(\gamma-1)\times(\alpha-\gamma+1)}F.$$

Proof. Direct verification.

References

1. M. Hall. "Combinatorial Theory," Waltham, Mass., 1967.

2. A. Hedayat and S. Y. Robert Li, "The Trade-off Method in the BIB Designs with Variable Support Sizes." Am. Stat. 7, 1277-1287, 1979.